VEHICLE DYNAMICS - LATERAL

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ARTICULATED VEHICLE MODEL

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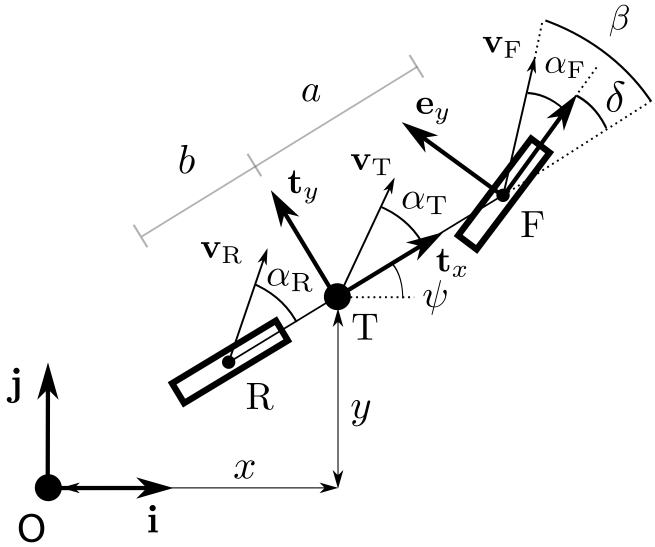

Figura 1 – Single track bicycle model.

0.1 USANDO LAGRANGE

O modelo físico do carro é ilustrado na figura 1. A base $\{\mathrm{O}\mathbf{ijk}\}$ é fixa no referencial inercial. A base $\{\mathrm{T}\mathbf{t}_x\mathbf{t}_y\mathbf{t}_z\}$ é solidária ao veículo. A base $\{\mathrm{F}\mathbf{e}_x\mathbf{e}_y\mathbf{e}_z\}$ é solidária ao eixo dianteiro. O ponto T localiza o centro de massa do veículo. F e R localizam os eixos dianteiro e traseiro, respectivamente. O ponto O é a origem do sistema e se encontra fixo no referencial inercial. A distância a separa os pontos F e T e a distância b separa os pontos T e R. a As coordenadas generalizadas são dadas por

$$q_1 = x \tag{1a}$$

$$q_2 = y \tag{1b}$$

$$q_3 = \psi, \tag{1c}$$

onde x e y são as coordenadas longitudinais do centro de massa do carro nas direções horizontal e vertical, respectivamente. ψ é o ângulo de orientação absoluta do veículo.

O vetor posição do centro de massa em relação ao ponto O é

$$\mathbf{p}_{\mathrm{T/O}} = x\mathbf{i} + y\mathbf{j}. \tag{2}$$

Derivando a equação (2) em relação ao tempo temos

$$\mathbf{v}_\mathrm{T} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}. \tag{3}$$

A energia cinética do sistema é

$$T = \frac{1}{2} m_T \mathbf{v}_\mathrm{T} \cdot \mathbf{v}_\mathrm{T} + \frac{1}{2} I_T \dot{\psi}^2. \tag{4}$$

Ou ainda,

$$T = \frac{1}{2} m_T \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} I_T \dot{\psi}^2. \tag{5}$$

A velocidade no eixo dianteiro é dada por

$$\mathbf{v}_\mathrm{F} = \mathbf{v}_\mathrm{T} + \mathbf{r} \wedge \mathbf{r}_{\mathrm{F/T}}, \tag{6}$$

onde $\mathbf{r}_{\mathrm{F/T}}$ é o vetor posição do ponto F em relação ao ponto T. Logo,

$$\mathbf{v}_\mathrm{F} = \left(\dot{x} - a\dot{\psi} \sin\psi \right) \mathbf{i} + \left(\dot{y} + a\dot{\psi} \cos\psi \right) \mathbf{j}. \tag{7}$$

A velocidade no eixo traseiro é

$$\mathbf{v}_\mathrm{R} = \mathbf{v}_\mathrm{T} + \mathbf{r} \wedge \mathbf{r}_{\mathrm{R/T}}, \tag{8}$$

onde $\mathbf{r}_{\mathrm{R/T}}$ é o vetor posição do ponto R em relação ao ponto T. Logo,

$$\mathbf{v}_\mathrm{R} = \left(\dot{x} + b\dot{\psi} \sin\psi \right) \mathbf{i} + \left(\dot{y} - b\dot{\psi} \cos\psi \right) \mathbf{j} \tag{9}$$

Utilizando as equações (7) e (9), os ângulos de deriva podem ser escritos como

$$\alpha_\mathrm{F} = \arctan \left(\frac{\dot{y} + a\dot{\psi} \cos\psi}{\dot{x} - a\dot{\psi} \sin\psi} \right) - (\delta + \psi) \tag{10a}$$

$$\alpha_\mathrm{R} = \arctan \left(\frac{\dot{y} - b\dot{\psi} \cos\psi}{\dot{x} + b\dot{\psi} \sin\psi} \right) - \psi \tag{10b}$$

A força do eixo dianteiro é dadas por

$$\mathbf{F}_\mathrm{F} = F_{x,\mathrm{F}} \mathbf{e}_x + F_{y,\mathrm{F}} \mathbf{e}_x, \tag{11}$$

que pode ser escrita como

$$\mathbf{F}_\mathrm{F} = \left[F_{x,\mathrm{F}} \cos(\psi + \delta) - F_{y,\mathrm{F}} \sin(\psi + \delta) \right] \mathbf{i} + \left[F_{x,\mathrm{F}} \sin(\psi + \delta) + F_{y,\mathrm{F}} \cos(\psi + \delta) \right] \mathbf{j}. \tag{12}$$

A força no eixo traseiro é

$$\mathbf{F}_\mathrm{R} = F_{x,\mathrm{R}}\mathbf{t}_x + F_{y,\mathrm{R}}\mathbf{t}_y \tag{13}$$

ou

$$\mathbf{F}_\mathrm{R} = \left[F_{x,\mathrm{R}}\cos\psi - F_{y,\mathrm{R}}\sin\psi\right]\mathbf{i} + \left[F_{x,\mathrm{R}}\sin\psi + F_{y,\mathrm{R}}\cos\psi\right]\mathbf{j}. \tag{14}$$

As forças generalizadas são

$$Q_k = \sum_{j=1}^{p} \mathbf{F}_j \cdot \frac{\partial \mathbf{p}_j}{\partial q_k} \tag{15}$$

No modelo

$$Q_1 = \mathbf{F}_\mathrm{F} \cdot \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_1} + \mathbf{F}_\mathrm{R} \cdot \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_1} \tag{16a}$$

$$Q_2 = \mathbf{F}_\mathrm{F} \cdot \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_2} + \mathbf{F}_\mathrm{R} \cdot \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_2} \tag{16b}$$

$$Q_3 = \mathbf{F}_\mathrm{F} \cdot \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_3} + \mathbf{F}_\mathrm{R} \cdot \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_3}, \tag{16c}$$

onde

$$\frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_1} = \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial x} = \mathbf{i} \tag{17a}$$

$$\frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_2} = \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial y} = \mathbf{j} \tag{17b}$$

$$\frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial q_3} = \frac{\partial \mathbf{p}_{\mathrm{F/O}}}{\partial \psi} = -a\sin\psi\mathbf{i} + a\cos\psi\mathbf{j} \tag{17c}$$

e

$$\frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_1} = \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial x} = \mathbf{i} \tag{18a}$$

$$\frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_2} = \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial y} = \mathbf{j} \tag{18b}$$

$$\frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial q_3} = \frac{\partial \mathbf{p}_{\mathrm{R/O}}}{\partial \psi} = b\sin\psi\mathbf{i} - b\cos\psi\mathbf{j} \tag{18c}$$

Substituindo as equações (12), (14), (17) e (18) nas equações (16) temos

$$Q_1 = F_{x,\mathrm{F}} \cos(\psi + \delta) + F_{x,\mathrm{R}} \cos\psi - F_{y,\mathrm{F}} \sin(\psi + \delta) - F_{y,\mathrm{R}} \sin\psi \tag{19a}$$

$$Q_2 = F_{x,\mathrm{F}} \sin(\psi + \delta) + F_{x,\mathrm{R}} \sin\psi + F_{y,\mathrm{F}} \cos(\psi + \delta) + F_{y,\mathrm{R}} \cos\psi \tag{19b}$$

$$Q_3 = F_{x,\mathrm{F}} a \sin\delta + F_{y,\mathrm{F}} a \cos\delta - F_{y,\mathrm{R}} b. \tag{19c}$$

A formulação de Lagrange é dada por

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k. \tag{20}$$

Para as três coordenadas generalizadas do sistema

$$\frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial x} = 0 \tag{21a}$$

$$\frac{\partial T}{\partial q_2} = \frac{\partial T}{\partial y} = 0 \tag{21b}$$

$$\frac{\partial T}{\partial q_3} = \frac{\partial T}{\partial \psi} = 0, \tag{21c}$$

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial T}{\partial \dot{x}} = m_T \dot{x} \tag{22a}$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial T}{\partial \dot{y}} = m_T \dot{y} \tag{22b}$$

$$\frac{\partial T}{\partial \dot{q}_3} = \frac{\partial T}{\partial \dot{\psi}} = I_T \dot{\psi}, \tag{22c}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m_T \ddot{x} \tag{23a}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m_T \ddot{y} \tag{23b}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = I_T \ddot{\psi}. \tag{23c}$$

Substituindo (23) e (21) em (20) temos as equações de movimento

$$m_T \ddot{x} = F_{x,\mathrm{F}} \cos(\psi + \delta) + F_{x,\mathrm{R}} \cos\psi - F_{y,\mathrm{F}} \sin(\psi + \delta) - F_{y,\mathrm{R}} \sin\psi \tag{24a}$$

$$m_T \ddot{y} = F_{x,\mathrm{F}} \sin(\psi + \delta) + F_{x,\mathrm{R}} \sin\psi + F_{y,\mathrm{F}} \cos(\psi + \delta) + F_{y,\mathrm{R}} \cos\psi \tag{24b}$$

$$I_T\ddot{\psi} = F_{x,\mathrm{F}}a\sin\delta + F_{y,\mathrm{F}}a\cos\delta - F_{y,\mathrm{R}}b, \tag{24c}$$

ou seja,

$$\ddot{x} = \frac{F_{x,\mathrm{F}}\cos(\psi+\delta) + F_{x,\mathrm{R}}\cos\psi - F_{y,\mathrm{F}}\sin(\psi+\delta) - F_{y,\mathrm{R}}\sin\psi}{m_T} \tag{25a}$$

$$\ddot{y} = \frac{F_{x,\mathrm{F}}\sin(\psi+\delta) + F_{x,\mathrm{R}}\sin\psi + F_{y,\mathrm{F}}\cos(\psi+\delta) + F_{y,\mathrm{R}}\cos\psi}{m_T} \tag{25b}$$

$$\ddot{\psi} = \frac{F_{x,\mathrm{F}}a\sin\delta + F_{y,\mathrm{F}}a\cos\delta - F_{y,\mathrm{R}}b}{I_T}. \tag{25c}$$

Os vetores de estado podem ser escolhidos como

$$\mathrm{z}_1 = x \tag{26a}$$

$$\mathrm{z}_2 = y \tag{26b}$$

$$\mathrm{z}_3 = \psi \tag{26c}$$

$$\mathrm{z}_4 = \dot{x} \tag{26d}$$

$$\mathrm{z}_5 = \dot{y} \tag{26e}$$

$$\mathrm{z}_6 = \dot{\psi} \tag{26f}$$

Logo, as equações de estado são

$$\dot{\mathrm{z}}_1 = \mathrm{z}_4 \tag{27a}$$

$$\dot{\mathrm{z}}_2 = \mathrm{z}_5 \tag{27b}$$

$$\dot{\mathrm{z}}_3 = \mathrm{z}_6 \tag{27c}$$

$$\dot{\mathrm{z}}_4 = \frac{F_{x,\mathrm{F}}\cos(\mathrm{z}_3+\delta) + F_{x,\mathrm{R}}\cos\mathrm{z}_3 - F_{y,\mathrm{F}}\sin(\mathrm{z}_3+\delta) - F_{y,\mathrm{R}}\sin\mathrm{z}_3}{m_T} \tag{27d}$$

$$\dot{\mathrm{z}}_5 = \frac{F_{x,\mathrm{F}}\sin(\mathrm{z}_3+\delta) + F_{x,\mathrm{R}}\sin\mathrm{z}_3 + F_{y,\mathrm{F}}\cos(\mathrm{z}_3+\delta) + F_{y,\mathrm{R}}\cos\mathrm{z}_3}{m_T} \tag{27e}$$

$$\dot{\mathrm{z}}_6 = \frac{F_{x,\mathrm{F}}a\sin\delta + F_{y,\mathrm{F}}a\cos\delta - F_{y,\mathrm{R}}b}{I_T} \tag{27f}$$

Com os ângulos de deriva, a partir das equações (10a) e (10b),

$$\alpha_\mathrm{F} = \arctan\left(\frac{\mathrm{z}_5 + a\mathrm{z}_6\cos\mathrm{z}_3}{\mathrm{z}_4 - a\mathrm{z}_6\sin\mathrm{z}_3}\right) - (\delta + \mathrm{z}_3) \tag{28}$$

$$\alpha_{\mathrm{R}} = \arctan\left(\frac{\mathrm{z}_5 - b\mathrm{z}_6\cos\mathrm{z}_3}{\mathrm{z}_4 + b\mathrm{z}_6\sin\mathrm{z}_3}\right) - \mathrm{z}_3 \tag{29}$$

0.2 SUBSTITUIÇÃO DOS ESTADOS

Em muitas ocasiões é conveniente fazer a substituição dos estados $\dot{x}$ e $\dot{y}$ por v_{T} e α_{T}. A relação entre estes pares de estados é

$$\dot{x} = v_{\mathrm{T}}\cos\left(\psi + \alpha_{\mathrm{T}}\right) \tag{30a}$$

$$\dot{y} = v_{\mathrm{T}}\sin\left(\psi + \alpha_{\mathrm{T}}\right). \tag{30b}$$

Derivando em relação ao tempo a equação (30) temos

$$\ddot{x} = \dot{v}_{\mathrm{T}}\cos\left(\psi + \alpha_{\mathrm{T}}\right) - v_{\mathrm{T}}\left(\dot{\psi} + \dot{\alpha}_{\mathrm{T}}\right)\sin\left(\psi + \alpha_{\mathrm{T}}\right) \tag{31a}$$

$$\ddot{y} = \dot{v}_{\mathrm{T}}\sin\left(\psi + \alpha_{\mathrm{T}}\right) + v_{\mathrm{T}}\left(\dot{\psi} + \dot{\alpha}_{\mathrm{T}}\right)\cos\left(\psi + \alpha_{\mathrm{T}}\right). \tag{31b}$$

Substituindo as equações (31) nas equações (25)

$$\dot{v}_{\mathrm{T}}\cos\left(\psi + \alpha_{\mathrm{T}}\right) - v_{\mathrm{T}}\left(\dot{\psi} + \dot{\alpha}_{\mathrm{T}}\right)\sin\left(\psi + \alpha_{\mathrm{T}}\right) = \frac{F_{x,\mathrm{F}}\cos\left(\psi + \delta\right) + F_{x,\mathrm{R}}\cos\psi - F_{y,\mathrm{F}}\sin\left(\psi + \delta\right) - F_{y,\mathrm{R}}\,\mathrm{si}}{m_T} \tag{32a}$$

$$\dot{v}_{\mathrm{T}}\sin\left(\psi + \alpha_{\mathrm{T}}\right) + v_{\mathrm{T}}\left(\dot{\psi} + \dot{\alpha}_{\mathrm{T}}\right)\cos\left(\psi + \alpha_{\mathrm{T}}\right) = \frac{F_{x,\mathrm{F}}\sin\left(\psi + \delta\right) + F_{x,\mathrm{R}}\sin\psi + F_{y,\mathrm{F}}\cos\left(\psi + \delta\right) + F_{y,\mathrm{R}}\,\mathrm{co}}{m_T} \tag{32b}$$

$$\ddot{\psi} = \frac{F_{x,\mathrm{F}}a\sin\delta + F_{y,\mathrm{F}}a\cos\delta - F_{y,\mathrm{R}}b}{I_T}. \tag{32c}$$

Reescrevendo e simplificando temos

$$\dot{v}_{\mathrm{T}} = \frac{F_{x,\mathrm{F}}\cos\left(\alpha_{\mathrm{T}} - \delta\right) + F_{x,\mathrm{R}}\cos\alpha_{\mathrm{T}} + F_{y,\mathrm{F}}\sin\left(\alpha_{\mathrm{T}} - \delta\right) + F_{y,\mathrm{R}}\sin\alpha_{\mathrm{T}}}{m_T} \tag{33a}$$

$$\dot{\alpha}_{\mathrm{T}} = \frac{-F_{x,\mathrm{F}}\sin\left(\alpha_{\mathrm{T}} - \delta\right) - F_{x,\mathrm{R}}\sin\alpha_{\mathrm{T}} + F_{y,\mathrm{F}}\cos\left(\alpha_{\mathrm{T}} - \delta\right) + F_{y,\mathrm{R}}\cos\alpha_{\mathrm{T}} - m_T v\dot{\psi}}{m_T v_{\mathrm{T}}} \tag{33b}$$

$$\ddot{\psi} = \frac{F_{x,\mathrm{F}}a\sin\delta + F_{y,\mathrm{F}}a\cos\delta - F_{y,\mathrm{R}}b}{I_T}. \tag{33c}$$

E os ângulos de deriva passam a ser

$$\alpha_{\mathrm{F}} = \arctan\left(\frac{v_{\mathrm{T}}\sin\left(\psi+\alpha_{\mathrm{T}}\right)+a\dot{\psi}\cos\psi}{v_{\mathrm{T}}\cos\left(\psi+\alpha_{\mathrm{T}}\right)-a\dot{\psi}\sin\psi}\right)-\left(\delta+\psi\right) \tag{34a}$$

$$\alpha_{\mathrm{R}} = \arctan\left(\frac{v_{\mathrm{T}}\sin\left(\psi+\alpha_{\mathrm{T}}\right)-b\dot{\psi}\cos\psi}{v_{\mathrm{T}}\cos\left(\psi+\alpha_{\mathrm{T}}\right)+b\dot{\psi}\sin\psi}\right)-\psi \tag{34b}$$

Simplificando, temos que

$$\alpha_{\mathrm{F}} = \arctan\left(\frac{v_{\mathrm{T}}\sin\alpha_{\mathrm{T}}+a\dot{\psi}}{v_{\mathrm{T}}\cos\alpha_{\mathrm{T}}}\right)-\delta \tag{35a}$$

$$\alpha_{\mathrm{R}} = \arctan\left(\frac{v_{\mathrm{T}}\sin\alpha_{\mathrm{T}}-b\dot{\psi}}{v_{\mathrm{T}}\cos\alpha_{\mathrm{T}}}\right) \tag{35b}$$

Portanto, os estados passam a ser

$$\mathrm{x}_1 = x \tag{36a}$$

$$\mathrm{x}_2 = y \tag{36b}$$

$$\mathrm{x}_3 = \psi \tag{36c}$$

$$\mathrm{x}_4 = v_{\mathrm{T}} \tag{36d}$$

$$\mathrm{x}_5 = \alpha_{\mathrm{T}} \tag{36e}$$

$$\mathrm{x}_6 = \dot{\psi}. \tag{36f}$$

O vetor de estados pode ser escrito como

$$\mathbf{x} = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \\ \mathrm{x}_4 \\ \mathrm{x}_5 \\ \mathrm{x}_6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \psi \\ v_{\mathrm{T}} \\ \alpha_{\mathrm{T}} \\ \dot{\psi} \end{bmatrix} \tag{37}$$

e as equações de estado

$$\dot{\mathrm{x}}_1 = \mathrm{x}_4\cos\left(\mathrm{x}_3+\mathrm{x}_5\right) \tag{38a}$$

$$\dot{\mathrm{x}}_2 = \mathrm{x}_4\sin\left(\mathrm{x}_3+\mathrm{x}_5\right) \tag{38b}$$

$$\dot{\mathrm{x}}_3 = \mathrm{x}_6 \tag{38c}$$

$$\dot{\mathrm{x}}_4 = \frac{F_{x,\mathrm{F}} \cos\left(\mathrm{x}_5 - \delta\right) + F_{x,\mathrm{R}} \cos \mathrm{x}_5 + F_{y,\mathrm{F}} \sin\left(\mathrm{x}_5 - \delta\right) + F_{y,\mathrm{R}} \sin \mathrm{x}_5}{m_T} \tag{38d}$$

$$\dot{\mathrm{x}}_5 = \frac{-F_{x,\mathrm{F}} \sin\left(\mathrm{x}_5 - \delta\right) - F_{x,\mathrm{R}} \sin \mathrm{x}_5 + F_{y,\mathrm{F}} \cos\left(\mathrm{x}_5 - \delta\right) + F_{y,\mathrm{R}} \cos \alpha_{\mathrm{T}} - m_T \mathrm{x}_4 \mathrm{x}_6}{m_T \mathrm{x}_4} \tag{38e}$$

$$\dot{\mathrm{x}}_6 = \frac{F_{x,\mathrm{F}} a \sin \delta + F_{y,\mathrm{F}} a \cos \delta - F_{y,\mathrm{R}} b}{I_T} \tag{38f}$$

0.3 LINEARIZAÇÃO

As equações não lineares em (38) podem ser escritas como

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}\right), \tag{39}$$

onde o vetor de estados é

$$\mathbf{x} = \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \\ \mathrm{x}_4 \\ \mathrm{x}_5 \\ \mathrm{x}_6 \end{bmatrix}. \tag{40}$$

e o vetor de entradas é

$$\mathbf{u} = \begin{bmatrix} \delta \\ F_{x,\mathrm{F}} \\ F_{x,\mathrm{R}} \\ F_{y,\mathrm{F}} \\ F_{y,\mathrm{R}} \end{bmatrix}. \tag{41}$$

A função vetorial $\mathbf{f}$ é

$$\mathbf{f} = \begin{bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \\ \mathrm{f}_4 \\ \mathrm{f}_5 \\ \mathrm{f}_6 \end{bmatrix} = \begin{bmatrix} \dot{\mathrm{x}}_1 \\ \dot{\mathrm{x}}_2 \\ \dot{\mathrm{x}}_3 \\ \dot{\mathrm{x}}_4 \\ \dot{\mathrm{x}}_5 \\ \dot{\mathrm{x}}_6 \end{bmatrix}, \tag{42}$$

A linearização deste sistema pode ser feita para um veículo se movimentando em linha reta com uma determinada velocidade $v > 0$. Neste caso, os estados no ponto de operação são dados por

$$\mathrm{x}_{1,op} = x_{op} = ? \tag{43a}$$

$$\mathrm{x}_{2,op} = y_{op} = ? \tag{43b}$$

$$\mathrm{x}_{3,op} = \psi_{op} = 0 \tag{43c}$$

$$\mathrm{x}_{4,op} = v_{\mathrm{T},op} = v_{\mathrm{T},0} \tag{43d}$$

$$\mathrm{x}_{5,op} = \alpha_{\mathrm{T},op} = 0 \tag{43e}$$

$$\mathrm{x}_{6,op} = \dot{\psi}_{op} = 0, \tag{43f}$$

OBS: Os estados x e y não influenciam a dinâmica do sistema.

Vetor do ponto de operação dos estados é

$$\mathbf{x}_{op} = \begin{bmatrix} \mathrm{x}_{1,op} \\ \mathrm{x}_{2,op} \\ \mathrm{x}_{3,op} \\ \mathrm{x}_{4,op} \\ \mathrm{x}_{5,op} \\ \mathrm{x}_{6,op} \end{bmatrix}. \tag{44}$$

O ponto de operação das entradas é

$$\delta_{op} = 0 \tag{45a}$$

$$F_{x,\mathrm{F},op} = 0 \tag{45b}$$

$$F_{x,\mathrm{R},op} = 0 \tag{45c}$$

$$F_{y,\mathrm{F},op} = 0 \tag{45d}$$

$$F_{y,\mathrm{R},op} = 0. \tag{45e}$$

O vetor do ponto de operação das entradas é

$$\mathbf{u}_{op} = \begin{bmatrix} \delta_{op} \\ F_{x,\mathrm{F},op} \\ F_{x,\mathrm{R},op} \\ F_{y,\mathrm{F},op} \\ F_{y,\mathrm{R},op} \end{bmatrix}. \tag{46}$$

A expansão da série de Taylor truncada nos termos de primeira ordem é dada por

$$\mathbf{f}_{lin}\left(\mathbf{x}, \mathbf{u}\right) = \mathbf{f}\left(\mathbf{x}_{op}, \mathbf{u}_{op}\right) + \nabla\mathbf{f}\left(\mathbf{x}_{op}, \mathbf{u}_{op}\right) \begin{bmatrix} \mathbf{x} - \mathbf{x}_{op} \\ \mathbf{u} - \mathbf{u}_{op} \end{bmatrix}. \tag{47}$$

onde

$$\mathbf{f}\left(\mathbf{x}_{op}, \mathbf{u}_{op}\right) = \begin{bmatrix} v_{\mathrm{T},0} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{48}$$

$$\nabla\mathbf{f} = \begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial \psi} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial \alpha_\mathrm{T}} & \frac{\partial f_1}{\partial \dot{\psi}} & \frac{\partial f_1}{\partial \delta} & \frac{\partial f_1}{\partial F_{x,\mathrm{F}}} & \frac{\partial f_1}{\partial F_{x,\mathrm{R}}} & \frac{\partial f_1}{F_{y,\mathrm{F}}} & \frac{\partial f_1}{\partial F_{y,\mathrm{R}}} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial \psi} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial \alpha_\mathrm{T}} & \frac{\partial f_2}{\partial \dot{\psi}} & \frac{\partial f_2}{\partial \delta} & \frac{\partial f_2}{\partial F_{x,\mathrm{F}}} & \frac{\partial f_2}{\partial F_{x,\mathrm{R}}} & \frac{\partial f_2}{F_{y,\mathrm{F}}} & \frac{\partial f_2}{\partial F_{y,\mathrm{R}}} \\
\frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial \psi} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial \alpha_\mathrm{T}} & \frac{\partial f_3}{\partial \dot{\psi}} & \frac{\partial f_3}{\partial \delta} & \frac{\partial f_3}{\partial F_{x,\mathrm{F}}} & \frac{\partial f_3}{\partial F_{x,\mathrm{R}}} & \frac{\partial f_3}{F_{y,\mathrm{F}}} & \frac{\partial f_3}{\partial F_{y,\mathrm{R}}} \\
\frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial \psi} & \frac{\partial f_4}{\partial v} & \frac{\partial f_4}{\partial \alpha_\mathrm{T}} & \frac{\partial f_4}{\partial \dot{\psi}} & \frac{\partial f_4}{\partial \delta} & \frac{\partial f_4}{\partial F_{x,\mathrm{F}}} & \frac{\partial f_4}{\partial F_{x,\mathrm{R}}} & \frac{\partial f_4}{F_{y,\mathrm{F}}} & \frac{\partial f_4}{\partial F_{y,\mathrm{R}}} \\
\frac{\partial f_5}{\partial x} & \frac{\partial f_5}{\partial y} & \frac{\partial f_5}{\partial \psi} & \frac{\partial f_5}{\partial v} & \frac{\partial f_5}{\partial \alpha_\mathrm{T}} & \frac{\partial f_5}{\partial \dot{\psi}} & \frac{\partial f_5}{\partial \delta} & \frac{\partial f_5}{\partial F_{x,\mathrm{F}}} & \frac{\partial f_5}{\partial F_{x,\mathrm{R}}} & \frac{\partial f_5}{F_{y,\mathrm{F}}} & \frac{\partial f_5}{\partial F_{y,\mathrm{R}}} \\
\frac{\partial f_6}{\partial x} & \frac{\partial f_6}{\partial y} & \frac{\partial f_6}{\partial \psi} & \frac{\partial f_6}{\partial v} & \frac{\partial f_6}{\partial \alpha_\mathrm{T}} & \frac{\partial f_6}{\partial \dot{\psi}} & \frac{\partial f_6}{\partial \delta} & \frac{\partial f_6}{\partial F_{x,\mathrm{F}}} & \frac{\partial f_6}{\partial F_{x,\mathrm{R}}} & \frac{\partial f_6}{F_{y,\mathrm{F}}} & \frac{\partial f_6}{\partial F_{y,\mathrm{R}}}
\end{bmatrix}. \tag{49}$$

Calculando as derivadas parciais,

$$\nabla \mathbf{f}\left(\mathbf{x}_{op}, \mathbf{u}_{op}\right) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_{\mathrm{T},0} & 0 & v_{\mathrm{T},0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m_T} & \frac{1}{m_T} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \frac{1}{m_T v_{\mathrm{T},0}} & \frac{1}{m_T v_{\mathrm{T},0}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a}{I_{\mathrm{T}}} & -\frac{b}{I_{\mathrm{T}}} \end{bmatrix} \tag{50}$$

Substituindo as equações (48) e (50) em (47) o sistema linearizado é

$$f_{1,lin} = \dot{x} = v_{\mathrm{T}} \tag{51a}$$

$$f_{2,lin} = \dot{y} = v_{\mathrm{T},0}\left(\psi + \alpha_{\mathrm{T}}\right) \tag{51b}$$

$$f_{3,lin} = \dot{\psi} = \dot{\psi} \tag{51c}$$

$$f_{4,lin} = \dot{v}_{\mathrm{T}} = \frac{F_{x,\mathrm{F}} + F_{x,\mathrm{R}}}{m_T} \tag{51d}$$

$$f_{5,lin} = \dot{\alpha}_{\mathrm{T}} = \frac{F_{y,\mathrm{F}} + F_{y,\mathrm{R}}}{m_T v_{\mathrm{T},0}} - \dot{\psi} \tag{51e}$$

$$f_{6,lin} = \ddot{\psi} = \frac{aF_{y,\mathrm{F}} - bF_{y,\mathrm{R}}}{I_T} \tag{51f}$$

É importante notar que se as forças longitudinais do pneu, $F_{x,\mathrm{F}}$ e $F_{x,\mathrm{R}}$, forem iguais a zero o estado v_{T} é constante.

0.4 INCLUINDO PNEU LINEAR

Linearizando o valor do dos ângulos de deriva em (35) no mesmo ponto de operação temos

$$\alpha_{\mathrm{F},lin} = \alpha_{\mathrm{T}} + \frac{a}{v_{\mathrm{T},0}}\dot{\psi} - \delta \tag{52a}$$

$$\alpha_{\mathrm{F},lin} = \alpha_{\mathrm{T}} - \frac{b}{v_{\mathrm{T},0}}\dot{\psi}. \tag{52b}$$

Supondo uma lei de força linear para os pneu temos

$$F_{y,\mathrm{F}} = -K_{\mathrm{F}}\alpha_{\mathrm{F}} = -K_{\mathrm{F}}\alpha_{\mathrm{T}} - \frac{aK_{\mathrm{F}}}{v_{\mathrm{T},0}}\dot{\psi} + K_{\mathrm{F}}\delta \tag{53a}$$

$$F_{y,\mathrm{R}} = -K_{\mathrm{R}}\alpha_{\mathrm{R}} = -K_{\mathrm{R}}\alpha_{\mathrm{T}} + \frac{bK_{\mathrm{R}}}{v_{\mathrm{T},0}}\dot{\psi} \tag{53b}$$

Substituir as equações (53) nas equações linearizadas em (51) temos

$$f_{1,lin} = \dot{x} = v_{\mathrm{T}} \tag{54a}$$

$$f_{2,lin} = \dot{y} = v_{\mathrm{T},0}\left(\psi + \alpha_{\mathrm{T}}\right) \tag{54b}$$

$$f_{3,lin} = \dot{\psi} = \dot{\psi} \tag{54c}$$

$$f_{4,lin} = \dot{v}_{\mathrm{T}} = \frac{F_{x,\mathrm{F}} + F_{x,\mathrm{R}}}{m_T} \tag{54d}$$

$$f_{5,lin} = \dot{\alpha}_{\mathrm{T}} = -\frac{K_{\mathrm{F}} + K_{\mathrm{R}}}{m_T v_{\mathrm{T},0}}\alpha_{\mathrm{T}} - \frac{m_T v_{\mathrm{T},0} + \frac{aK_{\mathrm{F}} - bK_{\mathrm{R}}}{v_{\mathrm{T},0}}}{m_T v_{\mathrm{T},0}}\dot{\psi} + \frac{K_{\mathrm{F}}}{m_T v_{\mathrm{T},0}}\delta \tag{54e}$$

$$f_{6,lin} = \ddot{\psi} = -\frac{aK_{\mathrm{F}} - bK_{\mathrm{R}}}{I_T}\alpha_{\mathrm{T}} - \frac{a^2K_{\mathrm{F}} + b^2K_{\mathrm{R}}}{I_T v_{\mathrm{T},0}}\dot{\psi} + \frac{aK_{\mathrm{F}}}{I_T}\delta \tag{54f}$$

Na forma matricial

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\hat{\mathbf{u}} \tag{55}$$

Ou ainda

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{v}_{\mathrm{T}} \\ \dot{\alpha}_{\mathrm{T}} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & v_{\mathrm{T},0} & 0 & v_{\mathrm{T},0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{K_{\mathrm{F}}+K_{\mathrm{R}}}{m_T v_{\mathrm{T},0}} & -\frac{m_T v_{\mathrm{T},0} + \frac{aK_{\mathrm{F}}-bK_{\mathrm{R}}}{v_{\mathrm{T},0}}}{m_T v_{\mathrm{T},0}} \\ 0 & 0 & 0 & 0 & -\frac{aK_{\mathrm{F}}-bK_{\mathrm{R}}}{I_T} & -\frac{a^2K_{\mathrm{F}}+b^2K_{\mathrm{R}}}{I_T v_{\mathrm{T},0}} \end{bmatrix} \begin{bmatrix} x \\ y \\ \psi \\ v_{\mathrm{T}} \\ \alpha_{\mathrm{T}} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{m_T} & \frac{1}{m_T} \\ \frac{K_{\mathrm{F}}}{m_T v_{\mathrm{T},0}} & 0 & 0 \\ \frac{aK_{\mathrm{F}}}{I_T} & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ F_{x,\mathrm{F}} \\ F_{x,\mathrm{R}} \end{bmatrix} \tag{56}$$

REFERÊNCIAS